Models of the Positive Compositional Truth

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In our talk we will present some recent results obtained jointly with Cezary Cieśliński and Mateusz Łełyk concerning models of the theories of positive truth.

By PT^- we mean a natural modification of CT^- (i.e. the theory of classical compositional arithmetical truth CT with no induction for the truth predicate) in which the classical compositional axioms of CT^- are replaced with a typed version of the KF's axioms, i.e. we take CT^- axioms without the compositional axioms for the negation but with e.g. the following clauses added:

1. $\forall s,t \ \Big(T(s \neq t) \equiv (s)^\circ \neq (t)^\circ\Big)$.
2. $\forall \phi,\psi \ \Big(T(\neg(\phi \wedge \psi)) \equiv T(\neg\phi) \vee T(\neg\psi)\Big)$.
3. $\forall \phi(v) \ \Big(T(\neg\exists v\phi(v)) \equiv \forall x \ T(\neg\phi(\underline{x}))\Big)$.

Here s,t denote closed arithmetical terms, ϕ,ψ are arithmetical sentences, $\phi(v)$ is an arithmetical formula with one variable and $\underline{x}$ denotes the x-th numeral.

PT^- is a fairly natural compositional theory of truth which is semantically conservative over PA as opposed to CT^-.

Some natural extensions of PT^- have another desirable property: while still proof-theoretically conservative over PA, they have a superexponential speed-up over the base theory. To this end it is enough to add the following internal induction principle to PT^-:

$$\forall \phi \ \Big(\Big(\forall x \, T\phi(x) \rightarrow T\phi(Sx)\Big) \longrightarrow \Big(T\phi(0) \rightarrow \forall x \, T\phi(x)\Big)\Big).$$

Which can be weakened further while still maintaining speed-up to the following axiom of the internal induction for total formulae:

$$\forall \phi \in \text{Tot} \ \Big(\Big(\forall x \, T\phi(x) \to T\phi(Sx)\Big) \longrightarrow \Big(T\phi(0) \to \forall x \, T\phi(x)\Big)\Big),$$

where Tot is the class of formulae $\phi(x)$ such that:

$$\forall x \ \ T\phi(x) \vee T\neg\phi(x).$$

It is now a natural question to ask whether PT^- extended with either of the above axioms is semantically conservative over PA. It turns out however, that adding whichever of these principles indeed restricts the class of models of PA which admit a truth predicate satisfying them.

It is then a natural question to ask whether there are natural compositional theories of truth which both have speed-up over PA and remain semantically conservative over this theory. One example of such a theory due to Mateusz Łełyk is WPT^- with the internal induction axiom — where WPT^- is a compositional theory of truth with the axioms modelled after weak rather than strong Kleene's logic, e.g.:

$$\forall \phi, \psi \ \ \Big(T(\phi \vee \psi) \equiv \big(T\phi \wedge T\psi\big) \vee \big(T\phi \wedge T\neg\psi\big) \vee \big(T\neg\phi \wedge T\psi\big)\Big).$$

All these facts taken together yield some (very partial) results to the effect that theories satisfying compositional axioms for the strong Kleene's logic are not definable in analogous theories satisfying weak Kleene's logic.

If time allows we will also presents some of our earlier results concerning models of conservative theories of truth and sketch the possible lines of further research.